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PROCEDURES FOR CAPITAL BUDGETING UNDER UNCERTAINTY

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PROCEDURES
FOR CAPITAL BUDGETING
UNDER UNCERTAINTY[1]

By Stewart C. Myers[2]

I. INTRODUCTION AND SUMMARY

The theory of capital budgeting under uncertainty is in a state of unusual flux, and none of the available techniques can be advertised as general. However, it is not difficult to identify two distinct approaches among the various proposals.

1. Vary the hurdle rate to adjust for project risk.-- The most widely adopted rule is to accept a project if its net discounted present value is positive. If project B yields a stream of expected returns $\bar{B}(t)$, and $\rho(B)$ is the cutoff rate, or "cost of capital" appropriate to projects with B's risk characteristics, then net present value (NPV) at time t = 0 is

$$\text{NPV} = \sum_{t=1}^{\infty} \frac{\bar{B}(t)}{(1+\rho(B))^t} + B(0). \tag{1}$$

The initial cash flow B(0) will typically be an investment, and thus negative.

This rule is usually justified by the objective of maximizing shareholders' wealth, as indicated by the market price of the firm's stock at the beginning of period t = 0. Accordingly, the cost of capital is de-

fined as the minimum expected rate of return on a project (with given risk characteristics) such that share price is increased by the project's adoption. The cost of capital is thus used as a hurdle rate, with the height of the barrier depending on the risk characteristics of the project compared with those of alternative investments open to shareholders.

It is easy to point out deficiencies in this "NPV approach" -- for instance, most authors are conspicuously vague about how to measure the hurdle rates appropriate to projects with different risk characteristics. The important point for our purposes, however, is that the NPV approach presumes projects to be risk-independent. That is, it presumes that the value of project B does not depend on the risk characteristics of the firm's existing assets, or of other investments the firm may undertake.

2. Treat capital budgeting as a problem of portfolio selection.-- The framework for portfolio selection originally presented by Markowitz [10] [11] is now well-known and widely accepted, although difficulties in assembling data and performing the required calculations have limited its use in practice. The similarity between the tasks of portfolio selection and capital budgeting has led Lintner, among others, to conclude that "the problem of determing the best capital budget of any given size is formally identical to the solution of a security portfolio analysis."[3] Several adaptations of portfolio selection techniques to capital budgeting problems have been presented.[4] Although the process of adaptation is not easy, the difficulties involved have been viewed as evidence that capital budgeting is simply a harder problem--not that the use of port-

folio selection is misconceived.

The portfolio selection approach is fundamentally different from use of a variable hurdle rate in that it presumes projects to be risk-interdependent. In Markowitz's framework, the value of a security to an investor depends not only on the expectation and variance of its return, but also on the covariances of its return with returns on other securities. By analogy, it is assumed that the value of a capital budgeting project depends on the covariances of its return with the returns to other activities which have been or may be undertaken by the firm.

I think it fair to say that the portfolio selection approach is generally considered superior by academic writers precisely because it deals with this presumed risk-interdependence. However, it has never been established that projects are risk-interdependent, assuming that the objective is to maximize share price.

This is the central question to be investigated in this paper. I propose that capital budgeting projects are risk-independent,[5] and thus that procedures using a variable hurdle rate are entirely appropriate from a theoretical standpoint.[6]

This paper is organized as follows. In Section II, I define risk-independence more precisely and set out the time-state-preference model of security valuation within which the argument is framed. In Section III, risk-independence is shown to be a necessary condition for security market equilibrium. Various special assumptions are required to reach this conclusion, but I do not regard them as unduly restrictive. A corollary

result is also established: that mergers undertaken solely for purposes of diversification cannot benefit all the shareholders of the merged firms. Section IV contains a comment on, and a discussion of possible objections to my proof. A summary of implications and topics for further research concludes the paper.

II. BACKGROUND AND DEFINITIONS

Defining Risk Independence

Consider the capital budgeting problem of a firm faced with the investment opportunities B and C. Assume that B and C are not mutually exclusive, that capital is not rationed, and that the firm is all-equity financed. The firm's objective is to maximize the sum of (1) ex-dividend share price at $t = 0$ and (2) the dividend paid at $t = 0$. We assume that the reductions in the current dividend required to finance projects A and/or B are known with certainty, so that the capital budgeting decision hinges on ex-dividend share price.

We will refer to the possible decisions as A, AB, AC and ABC, where:

A implies acceptance of neither project;
AB " " " B only;
AC " " " C only, and
ABC " " " both B and C.

The share prices contingent on the decisions are P_A, P_{AB}, P_{AC} and P_{ABC}.

The accept or reject decisions for projects B and C may be made independently if

$$P_{AB} - P_A = P_{ABC} - P_{AC} . \tag{2}$$

That is, independence requires that the increment in share price due to

acceptance of project B is not changed if C is also accepted.[7]

A necessary condition for Eq. (2) to hold is that the various alternatives are *physically* independent. That is, whether project B is adopted must not be one of the causal factors determining the actual cash flows generated in any future contingency by project C or the firm's present assets. The meaning of this condition may be better appreciated by thinking of the returns on the j^{th} and k^{th} securities, which are physically independent (from the investor's viewpoint) in just this sense. Although the securities' returns may be *statistically* related, the *actual* future prices and dividends of security j are not affected by whether the investor buys security k.

From this point on we will assume that all projects are physically independent.[8] Given this proviso, Eq. (2) becomes a sufficient condition for *risk*-independence. Note that there are no restrictions on the correlations among returns on B, C and the firm's existing assets, or on any other risk characteristics of the various options.

Security Valuation Under Uncertainty

As Eq. (2) indicates, risk-independence requires a particular relationship of a security's market price to the risk characteristics of its returns. Therefore, any proof or disproof must be based on a theory of security valuation. In the remainder of this section, the relevant characteristics of a time-state-preference model are presented. The model is discussed in detail in another paper.[9]

Consider the stream of uncertain returns $[\tilde{A}(1), \tilde{A}(2), \ldots, \tilde{A}(t), \ldots]$, where $\tilde{A}(t)$ is the dividend per share paid by a hypothetical firm

in period t if alternative A is taken. There are two ways to describe uncertainty about this stream. One is to estimate probability distributions in order to compute measures such as the mean and variance of $\tilde{A}(t)$ and the covariances of $\tilde{A}(t)$ with returns on other projects or securities.

The procedure used in this paper is to associate a particular dividend with each of a set of states of nature which may occur at specified times. Suppose that for t = 1 there are 100 possible states, indexed by s. Then the actual dividend in period t = 1 will be one element of the set $\left\{ A(1,1), \ldots, A(1,s), \ldots, A(1,100) \right\}$. This set describes the particular bundle of contingent payments (for t = 1) which the investor obtains when he purchases one share at t = 0.

It is not sufficient simply to say that "a state of nature is a particular sequence of events," since the concept "event" may be defined in a literally infinite number of ways. For instance, each of the following specifications of state (1,1) is reasonable in an appropriate context.

1. state (1,1) ⟶ GNP rises next year.
2. state (1,1) ⟶ GNP rises and GM raises its dividend.
3. state (1,1) ⟶ GNP rises, GM raises its dividend and faculty salaries at State U. rise by ten percent or more.

It is evident that cases 1, 2 and 3 reflect successively finer partitions of the "complete" set of event-sequences. For our purposes, it is sufficient that, for each period t, a state of nature can be associated with each possible sequence of returns on (1) available securities and (2) the projects B and C. We will assume that all investors have such a catalogue in mind.

Therefore, given that the contingency (s,t) occurs, we know the sequences of returns paid by the j^{th} and k^{th} securities: $R_j(s,1)$, $R_j(s,2)$, . . . , $R_j(s,t)$ and $R_k(s,1)$, $R_k(s,2)$, . . . , $R_k(s,t)$. State s + 1 might be associated with the same sequences of returns on all securities except k, for which $R_k(s+1,\tau) = R_k(s,\tau) + \$1.00$ for $\tau = 1, 2, . . . , t$. The catalogue of states exhausts all such permutations. Also, since we will consider only the time span from t = 1 to a distant horizon period at t = T, the set of possible state-time combinations is finite and exhaustive with respect to possible sequences of security returns.[10]

Given a security's contingent payments and the attitudes and strategies of investors, the risk of a security is inherent in the pattern of its returns across the possible future states of nature. The risk characteristics of a security, therefore, are those characteristics of this pattern which affect the value of a security to investors.[11]

Assume perfect capital markets, that taxes can be ignored, that short selling is prohibited, and that the value of a security can be expressed solely as a function of the size and risk characteristics of its stream of cash returns. Let securities be indexed by k and investors by i. Then the equilibrium of security markets requires that:[12]

$$P_k \geq \sum_{s,t} q_i(s,t)R_k(s,t), \tag{3}$$

for all i and all k, where

$$q_i(s,t) = \frac{\pi_i(s,t)U_i'(s,t)}{U_i'(0)} .$$

The variables used in Eq. (3) are defined as follows:

P_k = the equilibrium price of security k;

$R_k(s,t)$ = the cash return paid by the kth security in (s,t);

$\pi_i(s,t)$ = the ith investor's assessment of the probability that state (s,t) will occur;

$U_i'(s,t)$ = the marginal utility to the ith investor of income received in (s,t), and

$U_i'(0)$ = the marginal utility of current consumption to the ith investor.

The notation $\sum_{s,t}$ denotes summation over all states and all time periods from $t = 1$ to $t = T$.

A necessary condition for equilibrium is that each investor maximizes the expected utility of his portfolio, given the equilibrium security prices. Equation (3) states this condition formally: an investor's portfolio cannot be optimal unless the price of each security is at least equal to the marginal expected utility associated with a small increment in his holdings of that security, where the marginal utility of income in future contingencies is measured in terms of the marginal utility of current consumption. The term $q_i(s,t)$ indicates the present value to the ith investor of an incremental dollar of income which will be received at time t if and only if state (s,t) occurs. Similarly, the right hand side of Eq. (3) is a measure of the present value to the ith investor of one share of security k.

The value of any security depends on both the scale and the risk characteristics of its stream of returns. We will adjust for scale by dividing both sides of Eq. (3) by $S_k = \sum_{s,t} R_k(s,t)$. To simplify notation, the row vectors Q_i and X_k will be defined as the sets $\{q_i(s,t)\}$ and $\{R_k(s,t)/S_k\}$, respectively, written out in vector form. Each vector con-

tains one element for each state (s,t). Equation (3) may now be rewritten:

$$\mu_k = \frac{P_k}{S_k} \geq Q_i X_k' \quad . \tag{4}$$

The vector X_k describes the pattern across states of security k's returns; thus it indicates the security's risk characteristics.

Using this notation, the requirement for risk-independence is:

$$\mu_{AB}\, S_{AB} - \mu_A\, S_A = \mu_{ABC}\, S_{ABC} - \mu_{AC}\, S_{AC} \ . \tag{5}$$

Equations (3) and (4) are the most general, and thus the least precise, of the valuation formulas we will consider. Different assumptions yield more powerful formulas, of which the following will be useful here:[13]

1. If the kth security is included in the ith investor's optimal portfolio, then Eqs. (3) and (4) are equalities.

2. If short sales are feasible with no margin requirements or other restrictions,

$$\mu_k = Q_i X_k' \quad , \tag{6}$$

for all k and all i.

3. Let N and M denote the number of securities and states, respectively. Assume $N \geqslant M$, and that M of the vectors X_k are linearly independent. If short sales are not restricted, then the prices $\{q_i(s,t)\}$ will be identical for all investors, allowing substitution of a single vector Q in Eq. (6):

$$\mu_k = QX_k' \ . \tag{7}$$

In this special case, there is a definite equilibrium price for each possible contingent return.

III. A PROOF OF RISK-INDEPENDENCE

Analysis from a Single Investor's Point of View

We begin by supposing that the i^{th} investor is considering buying stock in the firm considering the projects B and C. If neither project is accepted, he can obtain a bundle of contingent returns $\{A(s,t)\}$ for each share bought--that is, a bundle of returns with scale S_A and risk characteristics described by the vector X_A. We know that Eq. (4) holds:

$$\mu_A \geq Q_i X_A' .$$

Since there are no physical interdependencies, the bundles of contingent returns associated with the decisions AB, AC and ABC are, respectively, $\{A(s,t) + B(s,t)\}$, $\{A(s,t) + C(s,t)\}$ and $\{A(s,t) + B(s,t) + C(s,t)\}$, where $B(s,t)$ and $C(s,t)$ are the incremental contingent cash flows of the projects B and C. For decision AB, it follows that:

$$S_{AB}\ X_{AB} = S_A\ X_A + S_B\ X_B$$

$$X_{AB} = \frac{S_A}{S_{AB}} X_A + \frac{S_B}{S_{AB}} X_B , \tag{8}$$

where $S_{AB} = S_A + S_B = \sum_{s,t} [A(s,t) + B(s,t)]$. Thus the vector X_{AB} which describes the risk characteristics of the contingent returns $\{A(s,t) + B(s,t)\}$ is a simple weighted average of the vectors X_A and X_B. The vectors X_{AC} and X_{ABC} are computed by the same procedure.

Now consider the increment to share value, from the i^{th} investor's point of view, if project B is adopted. If project C is not adopted, let $\Delta V\big|_A$ be the change in value:

$$\Delta V\big|_A = Q_i S_{AB} X'_{AB} - Q_i S_A X'_A = Q_i[(S_A X'_A + S_B X'_B) - S_A X'_A]$$

$$\Delta V\big|_A = Q_i S_B X'_B \ .$$

If project C is also adopted,

$$\Delta V\big|_{AC} = Q_i S_{ABC} X'_{ABC} - Q_i S_{AC} X'_{AC}$$

$$= Q_i[(S_A X'_A + S_B X'_B + S_C X'_C) - (S_A X'_A + S_C X'_C)],$$

$$\Delta V\big|_{AC} = Q_i S_B X'_B = \Delta V\big|_A \ .$$

Therefore the change in share value associated with project B does not depend on whether project C is also accepted. We might say that projects B and C are risk-independent from the point of view of any individual--although this does not prove Eq. (2) or Eq. (5).

Essentially the same point can be made in another way. Suppose the i^{th} investor holds the same proportion of the outstanding shares of two firms. Either firm can undertake project B, and the incremental contingent cash flows associated with B do not depend on which firm makes the investment. In this case, B's value to the investor does not depend on which firm accepts the project, regardless of any differences in the initial risk characteristics of the two firm's shares.

The Structure of Equilibrium Prices: A Special Case

These results suggest why risk independence is plausible, but they do not suffice as proof. Eq. (9) is a statement about what an individual would pay for marginal amounts of different bundles of contingent returns. But Eqs. (3) and (4) indicate that equilibrium does not, in general, constrain investors to agreement on the value of bundles of contingent returns. Therefore, we cannot infer the structure of equilibrium stock prices from analysis of the values of securities to any single investor.

As might be expected, risk-independence is easy to establish if investors agree on the values of contingent returns--that is, in the special case for which $Q_i = Q$, for all i, so that Eq. (7) can be used. Assuming this, and using Eq. (8),

$$\mu_{AB}S_{AB} - \mu_{A}S_{A} = Q[S_{AB}X'_{AB} - S_{A}X'_{A}] = QS_{B}X'_{B} \;.$$

$$\mu_{ABC}S_{ABC} - \mu_{AC}S_{ABC} = Q[S_{ABC}X'_{ABC} - S_{AC}X'_{AC}] = QS_{B}X'_{B} \;.$$

Therefore,

$$\mu_{AB}S_{AB} - \mu_{A}S_{A} = \mu_{ABC}S_{ABC} - \mu_{AC}S_{AC} \,,$$

which is Eq. (5), a sufficient condition for risk-independence.

This result could be questioned by criticizing the assumptions underlying it. It is not true, for instance, that there are as many securities as states of nature, or that there are no restrictions on short sales, as use of Eq. (7) implies. Nevertheless, if risk-independence is a necessary condition for equilibrium in this special case, then we would at least expect to find a tendency toward the same result in less perfect markets. Whether this tendency is blocked by market imperfections remains to be seen.

The Structure of Equilibrium Prices; General Case

From this point on the argument will be based on the more general condition for equilibrium:

$$\mu_k \geq Q_i X_k' , \tag{4}$$

for all i and all k. The use of Eq. (4) is appropriate if short sales are not permitted, and if the number of securities is less than the number of states. These conditions account for the possible inequality and for the fact that the vectors Q_i will vary among investors.

One more assumption is necessary: that there exist classes of "equivalent" securities, and that all investors agree on which securities belong to which classes. The classes are defined by the requirement that, given the vectors Q_i,[14]

$$Q_i X_j' = Q_i X_k' \tag{10}$$

for the "equivalent" securities j and k, and for all investors.[15] That is, if $S_j = S_k$, each investor is indifferent among holding one share of security j and one of k. If $S_j \neq S_k$, an investor would be willing to pay S_k/S_j times as much for k as for j. Note that this does not imply that all investors agree on the values of all securities, nor that an investor would be willing to purchase (marginal amounts of) each security at its equilibrium price.

This assumption has two important consequences, both of which are necessary to the proof of risk-independence. First, it obviously insures that $\mu_j = \mu_k$ if securities j and k are equivalent. Second, it constrains the relative equilibrium prices of securities in different classes.

Specifically, consider the security β with a pattern of contingent returns described by X_β. Security β is equivalent to a portfolio of other securities. That is,

$$Q_i X'_\beta = Q_i X'_P = Q_i \left[\sum_{k \neq \beta} w_\beta(k) X'_k \right], \tag{11}$$

for all i, where $w_\beta(k)$ are the weights required to obtain the portfolio P which is equivalent to β. If S_P is the scale of this portfolio's contingent returns, and S_{kP} is the part of S_P due to security k, then $w_\beta(k) = S_{kP}/S_P$. We assume $w_\beta(k) \geq 0$ for all k.[16]

If Eq. (11) holds, then a necessary condition for equilibrium is that

$$\mu_\beta = \sum_{k \neq \beta} w_\beta(k) \mu_k . \tag{12}$$

The proof is fairly simple. First, it is clear that μ_β cannot be greater than $\sum_{k \neq \beta} w_\beta(k) \mu_k$. Otherwise, no one would hold β.[17]

Suppose, on the other hand, that $\mu_\beta < \sum_{k \neq \beta} w_\beta(k) \mu_k$. It is shown in the appendix that in this case it will pay investors to sell other securities in order to purchase security β. Such trading is inconsistent with equilibrium; consequently, Eq. (12) must hold at equilibrium.

Now we can turn again to the decisions facing the hypothetical firm. Given the number of states which would have to be included in the set $\{(s,t)\}$, it is extremely unlikely that the possible patterns of contingent returns X_A, X_{AB}, X_{AC} and X_{ABC} would be identical to the patterns of other securities or portfolios. However, it would not be surprising to find that these patterns are equivalent to patterns which investors can

obtain.

The obvious thing is to distinguish among (1) cases in which investors can obtain patterns equivalent to X_A, X_B, and X_C by investing in other securities,[18] and (2) cases in which they cannot. If case (2) obtains, the projects may not be risk-independent. But they are in case (1).

If investors can obtain patterns of contingent returns equivalent to X_A, X_B and X_C, then

$$Q_i X'_j = Q_i \sum_{k \neq j} w_j(k) X'_k \,, \qquad j = A, B, C. \tag{13}$$

Now consider the patterns of contingent returns X^*_A, X^*_B and X^*_C, defined so that $X^*_j = \sum w_j(k) X_k$ and $S^*_j = S_j$, j = A, B, C. The reason for introducing the new variables is that if risk-independence holds for dummy projects with these characteristics, then it holds for the actual projects as well. To see this, note that X^*_A, X^*_B and X^*_C are equivalent to X_A, X_B and X_C, respectively. Further, since $X^*_{AB} = \frac{S_A}{S_{AB}} X^*_A + \frac{S_B}{S_{AB}} X^*_B$, we know that X^*_{AB} and X_{AB} are equivalent. A similar argument shows that X^*_{AC} and X^*_{ABC} are equivalent to X_{AC} and X_{ABC}, respectively. That the patterns are equivalent insures that $\mu^*_A = \mu'_A$, $\mu^*_B = \mu'_B$, and so on. Thus if Eq. (5) holds for the dummy projects, it holds for the actual projects also.

From Eq. (12), we know that

$$\mu_j = \sum w_j(k) \mu_k = \mu^*_j \qquad j = A, B, C. \tag{14}$$

Note that we can express X^*_{AB} as

$$X^*_{AB} = \frac{S_A}{S_{AB}} X^*_A + \frac{B}{S_{AB}} X^*_B = \sum_k \left[\frac{S_A}{S_{AB}} w_A(k) + \frac{S_B}{S_{AB}} w_B(k) \right] X_k . \tag{15}$$

Let

$$w_{AB}(k) = \frac{S_A}{S_{AB}} w_A(k) + \frac{S_B}{S_{AB}} w_B(k).$$

It follows from Eq. (12) that

$$\mu^*_{AB} = \sum_k w_{AB}(k) \mu_k .$$

The values of μ^*_{AC} and μ^*_{ABC} are computed in the same way.

Therefore,

$$\mu^*_{AB} S_{AB} - \mu^*_A S_A = S_{AB} \sum w_{AB}(k) \mu_k - S_A \sum w_A(k) \mu_k$$

$$= S_B \sum w_B(k) \mu_k .$$

$$\mu^*_{ABC} S_{ABC} - \mu^*_{AC} S_{AC} = S_{ABC} \sum w_{ABC}(k) \mu_k - S_{AC} \sum w_{AC}(k) \mu_k$$

$$= S_B \sum w_B(k) \mu_k$$

$$\mu^*_{AB} S_{AB} - \mu^*_A S_A = \mu^*_{ABC} S_{ABC} - \mu^*_{AC} S_{AC} . \tag{16}$$

Substituting μ_A for μ^*_A, μ_{AB} for μ^*_{AB}, etc., we have Eq. (5). This proves that projects B and C are risk-independent.

In summary, we have shown that if the feasible capital budgeting decisions all yield dividend streams with risk characteristics equivalent to those of returns obtainable via investment in securities, then the projects

under consideration are risk-independent.

A Corollary on Mergers

The discussion of risk-independence leads directly to a proposition about the value to investors of mergers undertaken for purposes of diversification. We will show that there is no net benefit due to diversification to the shareholders of the merged firms. Essentially the same point has been made by Professors Gort and Alberts at a symposium on "The Corporate Merger."[19] To the best of my knowledge, however, what follows is the first rigorous proof of the proposition.

Consider two firms J and K, with bundles of aggregate contingent dividends $\{J(s,t)\}$ and $\{K(s,t)\}$. We will continue to assume all-equity financing.

If diversification is the only motive for the merger, then X_{JK}, the vector describing the risk characteristics of the combined firms, is

$$X_{JK} = \frac{S_J}{S_{JK}} X_J + \frac{S_K}{S_{JK}} X_K, \tag{17}$$

where $S_J = \sum_{s,t} J(s,t)$ and $S_K = \sum_{s,t} K(s,t)$.

It is true that the actual risk characteristics of combined firms would rarely, if ever, be exactly as described by X_{JK} in Eq. (17). But the differences may not be material. In any case, they are not caused by diversification per se.[20] Net costs or benefits which occur because of such differences are attributable to physical interdependence between the operations of the merged firms, or possibly to other causes.[21]

By "no net benefit due to diversification" is meant that, if Eq. (17)

holds, then the total market value of the two firms is not increased by the merger. That is,

$$V_{JK} \leq V_J + V_K, \tag{18}$$

where V_{JK} is the total market value of the merged firms' stock, and V_J and V_K are the total market values of the stock of firms J and K, respectively, before announcement of the merger.

Equation (18) can be established by the following argument. First, we note from Eq. (17) that X_{JK} is in the cone spanned by X_J and X_K, and thus that before the merger investors can obtain portfolios, or segments of portfolios, with risk characteristics $X_P = X_{JK}$ regardless of whether the merger takes place. Thus the cost of obtaining such a portfolio is

$$\mu_P = \frac{S_J}{S_{JK}} \mu_J + \frac{S_K}{S_{JK}} \mu_K \, . \tag{19}$$

If we assume the patterns X_J and X_K are each equivalent to patterns obtainable by investing in other securities, then Eq. (12) implies that $\mu_{JK} = V_{JK}/S_{JK}$ is equal to μ_P. In this case Eq. (18) holds with an equality. However, an even more general argument can be made.

Assume that the patterns X_J and X_K are not equivalent to patterns obtainable by other means, and the prices of all other securities are fixed.[22] Suppose the merger is imposed on an existing equilibrium of security markets, and that $\mu_{JK} = \mu_P$, where μ_P is calculated from Eq. (19). The total market value of the firm can exceed $V_J + V_K$ only if there is an excess demand for securities of the merged firms at this price. Since the merger is the only change imposed on the pre-merger equilibrium, any excess demand can come only from investors already holding the stock of J,

K, or both. (That is, any demand for bundles of returns with risk characteristics X_{JK} at the "price" μ_{JK} is expressed by purchases of securities J and K prior to the merger.)

It is clear that the expected utility associated with these investors' portfolios cannot be increased by the merger. Otherwise they would have had an incentive to change their stockholdings in J and/or K beforehand, which contradicts the assumption of equilibrium. At the margin, therefore, the value to these investors of the stock of the merged firms must be less than or equal to the "price" μ_{JK}. That is,

$$(20) \qquad \mu_{JK} = \frac{V_{JK}}{S_{JK}} \geqslant Q_i X'_{JK},$$

which is simply a restatement of the necessary conditions for the i^{th} investor's equilibrium, as expressed by Eq. (3). If Eq. (20) holds with an equality for an investor, then his portfolio will be optimal both before and after the merger. Such an investor can contribute nothing to any excess demand. On the other hand, μ_{JK} may be greater than the right hand side of Eq. (20) for other investors, who can increase the expected utility of their portfolios by selling (at least) marginal amounts of the stock of the merged firm. At the assumed price, they will contribute to the excess supply of this stock. Since there cannot be any excess demand for the stock of the merged firm if $V_{JK} = V_J + V_K$, but there may be excess supply, the equilibrium price of the merged firm is at most equal to $V_J + V_K$.

IV. COMMENTS AND QUALIFICATIONS

This section is allocated to a general comment on the argument used to establish risk-independence, and to considering possible objections to this argument.

Comment: Risk-Independence and Modigliani and Miller's Proposition I.

Although it may not be evident immediately, the proof of risk-independence is logically similar to Modigliani and Miller's proof [13] of their well known Proposition I (that the market value of any firm is independent of financial leverage when corporate income is not taxed). The essence of this conclusion can be put this way:

> Proposition I:
> In an efficient market, the value of a pie is independent of how it is sliced.

The "pie" is the set of contingent cash flows which may be produced by the firm's assets; how it is sliced depends on the firm's choice among debt and equity financing.

The implications of risk-independence are analogous:

> Risk-independence:
> In an efficient market, a combination of pies will sell for no more or less than the sum of the pies' separate prices.

In this context, the pies are the investment projects open to the firm. If physical interdependence among projects is ruled out, the firm is no more or less than a collection of projects.

It should be clear that risk-independence and the irrelevance of financial leverage are two sides of the same coin. Each is presented as a necessary condition for equilibrium; consequently, the proof of each relies on the possibility of "arbitrage" among "equivalent" (combinations of) securities when the conditions for equilibrium are violated. In each case, the possibility of arbitrage confirms a more general point: that a corporation's financial decisions can benefit its shareholders only by providing the ability to construct portfolios with characteristics which would otherwise be impossible or more costly[24] to obtain.

Possible Qualifications

Several objections could be raised to the proof presented in Section III of this paper. For example, one could ask whether a partial equilibrium framework is really appropriate. In particular, is it reasonable to say that investors' evaluations of contingent returns, as measured by the vectors Q_i, are independent of the firm's capital budgeting decision?

Admittedly, it might not be reasonable if a firm's decisions forced its shareholders to accept large changes in the bundles of contingent returns yielded by their portfolios. Unfortunately, it is not possible to give a specific answer to the question, "How large is too large?" But note that the impact of an investment program that looks large to the corporate treasurer is diluted by several factors. First, to the extent that the investment program is anticipated[25] by investors, its impact on their assessments of the scale or risk characteristics of the firm's contingent dividends will be slight. Systematic replacement of obsolescent plant and equipment usually prompts little comment among investors, even if the amounts invested are substantial.[26]

A second factor diluting the impact of a firm's capital budgeting decision is the fact that most investors hold diversified portfolios, so that a given change in the bundle of contingent dividends a firm may pay will have a much smaller proportional effect on contingent portfolio returns. Moreover, even a non-diversified shareholder has the option of diversifying if the firm's capital budgeting decision puts him in an uncomfortable position. Even if he sells all of his shares, he is as well or better off as long as share price does not decline. But since share price is, in turn, related to all investors' demands, the fact that some port-

folios are relatively undiversified does not make a partial equilibrium framework inappropriate.

In short, a partial equilibrium framework may not be appropriate if (1) projects are large compared to the value of the firm's existing assets, (2) the scale and/or risk characteristics of the projects' incremental cash flows are unanticipated, (3) the firm's shareholders do not hold diversified portfolios, and (4) the shareholders are "locked in" to their holdings.[27] These conditions are plausible, perhaps, for proprietorships or small, closely-held corporations, but much less so for most publicly-owned firms.[28]

Essentially the same point can be made by noting that the vector Q_i reflects the investor's choice of a portfolio among thousands of available securities; it thus reflects the whole range of market opportunities. An investment program which appears very large in absolute terms may still be considered marginal in this market context.

A second possible objection is this: what if projects do not yield patterns of contingent returns which are equivalent to those obtainable by purchasing securities? This too is possible. But it is not clear that such projects will be risk-interdependent in any systematic way. This situation makes it difficult to prove that risk-independence holds precisely. It does not establish that the converse hypothesis holds in any meaningful sense.

Actually, assuming that projects have risk characteristics equivalent to (portfolios of) securities does not appear to be implausible for most practical purposes. In order to find a project violating this assumption, it would not be sufficient to find one with, say, a higher variance than

the projects a firm usually considers. Nor would it necessarily suffice to create a new product line or even a new industry. It would not be surprising to find that investors are indifferent among these projects' risk characteristics and those of _some_ existing securities.

Many other assumptions are required to prove that risk-independence holds precisely, few of which are "realistic" (the reader of any scholarly journal would be unrealistic if he expected them to be). Again, it is not apparent that the violation of any of these assumptions would lead to systematic risk-interdependence.

V. CONCLUSIONS

In this paper I have argued that risk-independence is a necessary consequence of the equilibrium of security markets under uncertainty. It is certainly possible that empirical tests will refute this prediction, despite the strong theoretical case that can be made for it. But the possible implications for financial management should be clear.

1. From the point of view of the firm which attempts to maximize its stock's market value, risk-independence implies that the risk characteristics of each investment project can be evaluated independently of the firm's other activities. Barring physical interdependencies--which admittedly pose important theoretical and practical problems--the optimal investment decision of the firm does not require complex decision procedures of the sort used in portfolio analysis. Specifying what this decision _does_ require is a topic that must be reserved for subsequent papers.

2. Diversification _per se_ is not an appropriate objective for firms which attempt to maximize their shareholders' wealth. This is true re-

gardless of whether diversification is achieved by merger or by internal expansion.

It should be emphasized, however, that risk-independence is perfectly consistent with the proposition that the value of a project depends on, say, the covariances of its returns with returns on other investment opportunities. What is implied is that the relevant "investment opportunities" are those facing the *investor*--i.e., the securities the investor can purchase. Risk-independence breaks down only if the risk characteristics of the *firm's* investment opportunities are not equivalent to those provided by *securities*.

Regardless of the plausibility of risk-independence on a theoretical level, a substantial empirical task remains. There is not only the specific question of whether projects can be considered risk-independent, but also the more general problem of determining the actual relationship between a security's risk characteristics and its market value. It is only prudent to anticipate a long wait before these questions are answered conclusively. In the meantime, I hope this paper prompts rethinking of the theory of capital budgeting under uncertainty.

APPENDIX

The problem is to show that the condition $\mu_\beta < \sum_{k \neq \beta} w_\beta(k)\mu_k$ cannot hold at equilibrium. Consider investor i who holds the portfolio Z with

$$X_Z = \sum_{k=1}^{N} \frac{S_{kZ}}{S_Z} X_k. \tag{21}$$

For each security oncluded in Z, $\mu_k = Q_i X_k'$. For the portfolio as a whole, therefore, we define

$$\mu_Z = \sum_{k=1}^{N} \frac{S_{kZ}}{S_Z} Q_i X_k' = \sum_{k=1}^{N} \frac{S_{kZ}}{S_Z} \mu_k. \tag{22}$$

Multiplying both sides by S_Z,

$$\mu_Z S_Z = \sum_{k=1}^{N} Q_i S_{kZ} X_k' \, . \tag{23}$$

That is, $\mu_Z S_Z$, the total market value of the portfolio, equals its present value to this investor at the "price" μ_Z.

Now the investor purchases, say, one hundred shares of β, obtaining a set of contingent returns with scale S_β. He sells $S_\beta w_\beta(k)$ shares of each security in portfolio Z. The result is a new portfolio Z*. Note that, since $\mu_\beta < \sum_{k \neq \beta} w_\beta(k)\mu_k$, he has cash left over.

Barring short sales, these transactions are feasible only if the original portfolio Z contains some shares of all securities for which $w_\beta(k) \geq 0$--i.e., only if we can use securities from Z to construct a portfolio with a pattern of contingent returns X_P. We will make this assumption.[29]

The present value to this investor of the new portfolio Z* is

(24) $$Q_iS_ZX'_{Z^*} = Q_iS_ZX'_Z + Q_iS_\beta\,[X'_\beta - \sum_{k\neq\beta} w_\beta(k)X'_k]$$

From Eq. (11),

$$Q_i[X'_\beta - \sum_{k\neq\beta} w_\beta(k)X'_k] = 0,$$

so that

(25) $$Q_iS_ZX'_{Z^*} = Q_iS_ZX'_Z\ .$$

This establishes that portfolios Z and Z^* are equally valuable to this investor. But since he can exchange Z^* for Z and have cash left over, it will obviously pay him to do so. Thus equilibrium cannot exist if $\mu_\beta < \sum_{k\neq\beta} w_\beta(k)\mu_k$. This completes the proof of Eq. (12).

FOOTNOTES

1. This paper is a further development of part of my doctoral dissertation, which was submitted to the Graduate School of Business, Stanford University, in 1967. I am indebted for good advice and apt suggestions to my dissertation committee, Professors Alexander Robichek, Gert von der Linde and Ezra Solomon. Also, Professor Kenneth Arrow was kind enough to read and comment on the entire dissertation. I wish also to thank Jack Hirshleifer, Avraham Beja and Peter Diamond for helpful comments.

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2. Assistant Professor of Finance, Sloan School of Management, Massachusetts Institute of Technology.

3. Lintner [7], p. 65. The italics are mine. See also his articles [8] [9] and Weingartner [21], pp. 499-504, esp. p. 503.

4. For example, Van Horne [20], Lintner [8] [9], Quirin [16] and Weingartner [21].

5. Michaelson has recognized the possibility of risk independence. See [12], p. 249.

6. Robichek and Myers [17] point out that use of a single discount rate is not always appropriate. A more general procedure is to discount the certainty equivalents of the expected returns at a riskless interest rate. The main point of this paper follows regardless of how NPV is calculated, however, so I have used the more familiar procedure exclusively.

7. Equation (2) also implies that $P_{AC} - P_A = P_{ABC} - P_{AB}$.

8. Note that this rules out mutually exclusive projects, and assumes that capital is not rationed. There is no doubt that these and other types of physical interdependencies are often important in practice. For a survey of work on these problems, see Weingartner [21].

9. Myers [15]. This and my dissertation [14] are extensions of a model proposed by Arrow [2] and developed by Hirshleifer [5] [6] and Debreu [3].

10. The horizon t = T is introduced for analytical convenience. Any errors due to not considering events subsequent to the horizon may be considered negligible if the horizon is far enough distant in time.

11. This way of defining "risk characteristic" is developed further in Robichek and Myers [19].

12. The proof is given in Myers [15].

13. See Myers [14] for proofs and discussion of these special cases.

14. The following proofs require only that Eq. (10) holds at the margin. It is not assumed that securities in an equivalent class are perfect substitutes in all situations. This is in contrast to the "equivalent return" classes postulated by Modigliani and Miller [13]. Further, Eq. (10) does not imply that returns on securities j and k are perfectly correlated, nor does it require a classification of securities by industry groupings.

    However, adopting the weaker concept of equivalence defined by Eq. (10) does not change Modigliani and Miller's conclusions regarding optimal capital structure for corporations. See Robichek and Myers [18].

15. The assumption that investors agree on which securities belong to which classes is a necessary consequence of equilibrium if there are no margin requirements for short selling. In this case, agreement on classes of equivalent securities follows directly from Eq. (6). I wish to thank Peter Diamond for pointing this out.

16. Thus short sales and borrowing are ruled out. This is not restrictive --the following proof is not changed significantly if negative weights are allowed.

17. It seems reasonable to ignore transaction costs at the present level of abstraction.

18. That is, securities other than the shares of our hypothetical firm.

19. Gort [4] and Alberts [1].

20. It should be clear that the term "diversification" is defined more restrictively here than in common usage.

21. Diversification and the associated smoothing of the firm's cash flows might allow economies in financing--e.g., the reduction of transaction costs if fewer financing operations are required.

22. If other prices vary, then we face a problem of dynamics for which the present model is unsuited. But I have not been able to conceive of any dynamic mechanism which systematically contravenes the results presented here. Certainly arguments in a static framework are used to justify applying portfolio analysis techniques to capital budgeting problems, or to justify diversification as a merger motive.

23. Avraham Beja has pointed out that a merger undertaken to diversify may reduce the welfare of investors, as judged by the criterion of Pareto optimality. This follows because no investors can be made better off, but some are likely to be worse off.

24. It is true that there are transaction costs associated with any portfolio shift. But the firm often faces similar costs in changing its portfolio of assets. The relevant question, therefore, is whether it is more costly for an investor to assemble a particular portfolio than it is for the firm to do it for him. This is not clear *a priori*. Would the per-share costs incurred by a firm in merging with another be less than those incurred by a stockholder who simply purchased shares of the acquired firm through his broker?

25. That is, to the extent it is consistent with the strategy that investors expect the firm to follow.

26. The decision *not* to replace would probably have a much greater impact --note fn. 25 above.

27. For example, shareholders may become "locked in" if they would incur a capital gain tax by selling out.

28. It is ironic that such firms are the most likely customers for portfolio analysis techniques for use in capital budgeting.

29. It is not as restrictive as it looks, since there may be other portfolios which also satisfy Eq. (21). For example, suppose that $w_\beta(k) > 0$. but that security j is in the same class as k. Then j could be substituted for k to form a new portfolio which would also be equivalent to β. Many combinations of securities j and k are equally acceptable. Since still other substitutions may be possible, Eq. (21) may hold for many different bundles of securities. The proof of Eq. (12) requires only that one of these bundles is a subset of a portfolio Z held by some investor.

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